Int. J. Solids Structures Vol. 16, pp. 337–345
Pergamon Press Ltd., 1980. Printed in Great Britain

SOME STATIC PROBLEMS OF MICROPOLAR CONTINUA†

J. P. NOWACKI and P. G. GLOCKNER
Department of Mechanical Engineering, The University of Calgary, Calgary, Alberta, Canada T2N 1N4

(*Received* 18 *December* 1978; *in revised form* 13 *April* 1979)

Abstract—In this paper it is shown that the solution, $(\bar{u}, \bar{\phi})$, to the nonhomogeneous equations of equilibrium for micropolar continua can be obtained by superposition of two parts, namely $(\bar{u}')$ and $(\bar{u}'', \bar{\phi})$. The first part, $\bar{u}'$, is identical to that obtained in the Hookean case, while the second part depends only on the constants α, β, γ and ϵ which are specific to Cosserat continua. Using these results and Fourier's integral transforms, Green's functions for a Cosserat body are obtained. In addition, solutions for the centre of compression and center of torsion are given and problems involving the distortions of micropolar continua are considered.

1. BASIC SOLUTION

Schaefer[1]‡ has shown that the solution to the homogeneous equations of equilibrium for a micropolar continuum can be constructed by superposition of two parts, the first of which is the same as for a Hookean body while the second part depends only on the characteristic Cosserat constants α, β, γ and ϵ. In this paper the same approach is used to obtain the solution to the corresponding nonhomogeneous equations of equilibrium from which Green's functions are derived. Similar results were obtained for the static case by Sandru[2] using Papkovitch's representation. Subsequently, Nowacki[3, 4] obtained Green's functions for the dynamic and thermodynamic case by means of Helmholtz decomposition.

The well-known Navier equations for elastic micropolar media are written as

$$D_1\bar{u} + (\lambda + \mu - \alpha)\,\text{grad div}\,\bar{u} + 2\alpha\,\text{curl}\,\bar{\phi} + \bar{X} = 0 \tag{1.1}$$

$$D_2\bar{u} + (\beta + \gamma - \epsilon)\,\text{grad div}\,\bar{\phi} + 2\alpha\,\text{curl}\,\bar{u} + \bar{Y} = 0 \tag{1.2}$$

where

$$D_1 = (\mu + \alpha)\nabla^2; \quad D_2 = (\gamma + \epsilon)\nabla^2 - 4\alpha.$$

Introduce $\bar{\xi}$ by

$$\bar{\xi} = \frac{1}{2}\,\text{curl}\,\bar{u} - \bar{\phi} \tag{1.3}$$

where (1/2) curl $\bar{u}$ represents the macrorotation, while $\bar{\phi}$ denotes the microrotation in the body.

Inserting (1.3) into (1.1) and (1.2), one arrives at

$$\mu\nabla^2\bar{u} + (\lambda + \mu)\,\text{grad div}\,\bar{u} + \bar{X} = 2\alpha\,\text{curl}\,\bar{\xi} \tag{1.4}$$

$$D_2\bar{\xi} + (\beta + \gamma - \epsilon)\,\text{grad div}\,\bar{\xi} - \bar{Y} = \frac{1}{2}(\gamma + \epsilon)\nabla^2\,\text{curl}\,\bar{u}. \tag{1.5}$$

Note, that the L.H.S. of eqn (1.4) has the same form as Navier's equation for the Hookean

†Results presented here were obtained in the course of research sponsored by the National Research Council of Canada, Grant No. A-2736.

‡Numbers in square brackets indicate publications listed under References.

body. Assume the solution to eqns (1.4) and (1.5) for the displacement field in the form

$$\bar{u} = \bar{u}' + \bar{u}'' \tag{1.6}$$

where $\bar{u}'$ is the solution to the equation

$$\mu\nabla^2\bar{u}' + (\lambda + \mu)\,\text{grad div}\,\bar{u}' + \bar{X} = 0. \tag{1.7}$$

Performing the curl operation on eqn (1.7), one obtains

$$\nabla^2\,\text{curl}\,\bar{u}' = \frac{1}{\mu}\,\text{curl}\,\bar{X}. \tag{1.8}$$

Substituting (1.6) and (1.7) into (1.4) and (1.5), and using (1.8), there result

$$\mu\nabla^2\bar{u}'' + (\lambda + \mu)\,\text{grad div}\,\bar{u}'' = 2\alpha\,\text{curl}\,\bar{\xi} \tag{1.9}$$

$$D_2\bar{\xi} + (\beta + \gamma - \epsilon)\,\text{grad div}\,\bar{\xi} - \bar{Y} = \frac{1}{2}(\gamma + \epsilon)\nabla^2\,\text{curl}\,\bar{u}'' - \frac{1}{2\mu}(\gamma + \epsilon)\,\text{curl}\,\bar{X}. \tag{1.10}$$

Performing the operations of divergence and curl, separately, on (1.9), leads to the following relations

$$\nabla^2\,\text{div}\,\bar{u}'' = 0; \quad \mu\nabla^2\bar{u}'' = 2\alpha\,\text{curl}\,\bar{\xi}. \tag{1.11a,b}$$

Taking the curl of (1.10) results in

$$D_2\,\text{curl}\,\bar{\xi} = \frac{1}{2}(\gamma + \epsilon)\nabla^2\,\text{curl curl}\,\bar{u}'' - \frac{1}{2\mu}(\gamma + \epsilon)\,\text{curl curl}\,\bar{X} + \text{curl}\,\bar{Y}. \tag{1.12}$$

Substituting eqn (1.11b) into the L.H.S. of this equation, after some algebra, one arrives at

$$\nabla^2 H\bar{u}'' = \frac{\alpha}{\mu(\mu + \alpha)}(\nabla^2 - \text{grad div})\bar{X} + \frac{1}{2\mu l^2}\,\text{curl}\,\bar{Y} \tag{1.13}$$

where

$$l^2 = \frac{(\mu + \alpha)(\gamma + \epsilon)}{4\alpha\mu}, \quad H = \nabla^2 - \frac{1}{l^2}.$$

The function $\bar{\phi}$ can be found directly from eqns (1.1) and (1.2). Performing the curl operation on (1.1) and the divergence on (1.2), leads to

$$D_2\,\text{curl}\,\bar{u} + 2\alpha\,\text{curl curl}\,\bar{\phi} + \text{curl}\,\bar{X} = 0 \tag{1.14a}$$

$$D_3\,\text{div}\,\bar{\phi} = -\text{div}\,\bar{Y} \tag{1.14b}$$

where

$$D_3 = [(\beta + 2\gamma)\nabla^2 - 4\alpha].$$

Applying D_2D_3 to eqn (1.2) and keeping in mind the relations (1.14), one obtains

$$DH\nabla^2\bar{\phi} = \frac{1}{4\alpha\mu l^2}\left[2\alpha D\,\text{curl}\,\bar{X} - (\mu + \alpha)\nabla^2 D\bar{Y} + (\mu + \alpha)\left(D - \frac{\gamma + \epsilon}{\beta + 2\gamma}H\right)\text{grad div}\,\bar{Y}\right] \tag{1.15}$$

where

$$D^2 = \nabla^2 - \frac{1}{\nu^2}, \quad \nu^2 = \frac{\beta + 2\gamma}{4\alpha}.$$

Equation (1.7) is solved using a similar procedure, to obtain

$$\nabla^2\nabla^2\bar{u}' = -\frac{1}{\mu}\left(\nabla^2\bar{X} - \frac{\lambda+\mu}{\lambda+2\mu}\,\text{grad div}\,\bar{X}\right). \tag{1.16}$$

Next, introduce Fourier transforms[5] as

$$\bar{u}_j(\bar{\zeta}) = \frac{1}{(2\pi)^{3/2}}\int\int_{-\infty}^{\infty}\int u_j(\bar{x})\exp(i\zeta_k x_k)\,\mathrm{d}V(\bar{x}) \tag{1.17}$$

$$u_j(\bar{x}) = \frac{1}{(2\pi)^{3/2}}\int\int_{-\infty}^{\infty}\int u_j(\bar{\zeta})\exp(-i\zeta_k x_k)\,\mathrm{d}W(\bar{\zeta}) \tag{1.18}$$

where

$$\mathrm{d}V(\bar{x}) = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3,\quad \mathrm{d}W(\bar{\zeta}) = \mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\,\mathrm{d}\zeta_3.$$

Using these transforms results in the solution to the problem at hand in the form

$$u_j'(\bar{x}) = \frac{1}{(2\pi)^{3/2}\mu}\int\int_{-\infty}^{\infty}\int \frac{\bar{X}_j}{\zeta^2}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) + \frac{\lambda+\mu}{(2\pi)^{3/2}\mu(\lambda+2\mu)}\frac{\partial^2}{\partial x_j\partial x_p}\int\int_{-\infty}^{\infty}\int\frac{\bar{X}_p}{\zeta^4}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) \tag{1.19}$$

$$\begin{aligned} u_j''(\bar{x}) = &-\frac{\alpha}{(2\pi)^{3/2}\mu(\mu+\alpha)}\int\int_{-\infty}^{\infty}\int\frac{\bar{X}_j}{\zeta^2+\frac{1}{l^2}}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) \\ &-\frac{\alpha}{(2\pi)^{3/2}\mu(\mu+\alpha)}\frac{\partial^2}{\partial x_j\partial x_p}\int\int_{-\infty}^{\infty}\int\frac{\bar{X}_p}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) \\ &+\frac{\epsilon_{jps}}{(2\pi)^{3/2}2\mu l^2}\frac{\partial}{\partial x_p}\int\int_{-\infty}^{\infty}\int\frac{\bar{Y}_s}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) \end{aligned} \tag{1.20}$$

$$\begin{aligned} \phi_j''(\bar{x}) = \frac{1}{4\alpha\mu l^2}\Bigg\{&\frac{2\alpha\epsilon_{jps}}{(2\pi)^{3/2}}\frac{\partial}{\partial x_p}\int\int_{-\infty}^{\infty}\int\frac{\bar{X}_s}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) \\ &+\frac{\mu+\alpha}{(2\pi)^{3/2}}\int\int_{-\infty}^{\infty}\int\frac{\bar{Y}_j}{\zeta^2+\frac{1}{l^2}}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) + \frac{\mu+\alpha}{2\pi}\frac{\partial^2}{\partial x_j\partial x_p} \\ &\times\int\int_{-\infty}^{\infty}\int\frac{\bar{Y}_p}{\zeta^2\left(\zeta^2+\frac{1}{l^2}\right)}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta}) - \frac{(\mu+\alpha)(\gamma+\epsilon)}{(2\pi)^{3/2}(\beta+2\gamma)}\frac{\partial^2}{\partial x_j\partial x_p} \\ &\times\int\int_{-\infty}^{\infty}\int\frac{\bar{Y}_p}{\zeta^2\left(\zeta^2+\frac{1}{\nu^2}\right)}(\mathrm{e}^{-ix_k\zeta_k})\,\mathrm{d}W(\bar{\zeta})\Bigg\}. \end{aligned} \tag{1.21}$$

One should note that every solution $(\bar{u}, \bar{\phi})$ of Navier's equations for micropolar media, eqns (1.1) and (1.2), may be obtained by superimposing two solutions; the first part, $\bar{u}'$ being the same as for the Hookean body, while the second part, $(\bar{u}'', \bar{\phi})$ is peculiar to Cosserat media. Therefore, if a solution is to be found to a Cosserat problem, described by relations (1.1) and (1.2) with known forces $\bar{X}$ and $\bar{Y}$, one can construct such a solution from the well known solution for Hooke's body, and add to that solution the additional terms, $\bar{u}''$ and $\bar{\phi}$, representing the Cosserat contribution.

2. GREEN'S FUNCTIONS

Using the results of the previous section, Green's functions for the displacement and the rotation for isotropic elastic micropolar media subjected to concentrated body forces and concentrated body couples are determined.

(a) *Concentrated body force*

Assume that a concentrated force parallel to the x_n-axis is applied at the origin of the coordinate system and that the body couples vanish, i.e.

$$X_j^{(n)} = \delta_3(\bar{x})\delta_{jn}, \quad Y_j = 0 \tag{2.1}$$

then

$$X_j^{(n)} = \frac{1}{(2\pi)^{3/2}} \int\int_{-\infty}^{\infty}\int \delta_{jn}\delta_3(\bar{x})(e^{-ix_k\zeta_k})\, dV(\bar{x}) = \frac{\delta_{jn}}{(2\pi)^{3/2}} \tag{2.2}$$

and eqns (1.19)–(1.21) take the form

$$u_j^{\prime(n)}(\bar{x}) = \frac{\delta_{jn}}{8\pi^3\mu} I_1 + \frac{\lambda+2\mu}{8\pi^3\mu(\lambda+2\mu)}\frac{\partial^2}{\partial x_n \partial x_j} I_2, \tag{2.3}$$

$$u_j^{\prime\prime(n)} = -\frac{\delta_{jn}\alpha}{8\pi^3\mu(\mu+\alpha)} I_3 - \frac{\alpha}{8\pi^3\mu(\mu+\alpha)}\frac{\partial^2}{\partial x_n \partial x_j} I_4, \tag{2.4}$$

$$\phi_j^{(n)}(\bar{x}) = \frac{\epsilon_{jnl}}{16\mu l^2\pi^3}\frac{\partial}{\partial x_l} I_4 \tag{2.5}$$

where

$$I_1 = \int\int_{-\infty}^{\infty}\int \frac{e^{-ix_k\zeta_k}}{\zeta^2}\, dW = \frac{2\pi^2}{R}$$

$$I_2 = \int\int_{-\infty}^{\infty}\int \frac{e^{-ix_k\zeta_k}}{\zeta^4}\, dW = -\pi^2 R$$

$$I_3 = \int\int_{-\infty}^{\infty}\int \frac{e^{-ix_k\zeta_k}}{\zeta^2 + \frac{1}{l^2}}\, dW = 2\pi^2\frac{e^{-R/l}}{R}$$

$$I_4 = \int\int_{-\infty}^{\infty}\int \frac{e^{-ix_k\zeta_k}}{\zeta^2\left(\zeta^2 - \frac{1}{l^2}\right)}\, dW = 2\pi^2 l^2 \frac{e^{-R/l} - 1}{R}$$

$$R = [(x_i - x_i')(x_i - x_i')]^{1/2}.$$

Substituting the above integrals into eqns (2.3)–(2.5) and using eqn (1.6), we obtain expressions for the displacement and rotation fields due to a concentrated force given by eqn (2.1). The solution to this problem shall be referred to as Green's functions for the case of a concentrated force, and is given as

$$G_{jn}^X(\bar{x}) = \frac{1}{8\pi\mu}\left(\delta_{jn}\nabla^2 R - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_n\nabla_j R\right) + B\left(l^2\nabla_j\nabla_n \frac{e^{-R/l}-1}{R} - \delta_{jn}\frac{e^{-R/l}}{R}\right), \tag{2.6}$$

$$\Phi_{jn}^X(\bar{x}) = \frac{1}{8\pi\mu}\epsilon_{njm}\nabla_m \frac{1-e^{-R/L}}{R} \tag{2.7}$$

where

$$B = \frac{\alpha}{4\pi\mu(\alpha+\mu)}$$

(b) *Concentrated body couple*

Assume that a concentrated couple, parallel to the x_n-axis is applied at the origin of the coordinate system, and that the body forces vanish, i.e.

$$X_j^{(n)} = 0, \quad Y_j^{(n)} = \delta_3(\bar{x})\delta_{jn} \tag{2.8}$$

then

$$\bar{Y}_j^{(n)} = \frac{\delta_{jn}}{(2\pi)^{3/2}}. \tag{2.9}$$

Equations (1.19)–(1.21) now take the form

$$u_j'^{(n)}(\bar{x}) = 0 \tag{2.10}$$

$$u_j''(\bar{x}) = \frac{\epsilon_{jln}}{16\mu l^2\pi^3}\nabla_l I_4 \tag{2.11}$$

$$\phi_j^{(n)}(\bar{x}) = \frac{1}{4\alpha\mu l^2}\left[\frac{\mu+\alpha}{8\pi^3}(\delta_{jn}I_3 + \nabla_j\nabla_n I_4) + \frac{(\mu+\alpha)(\gamma+\epsilon)}{8\pi^3(\beta+2\gamma)}\nabla_j\nabla_n I_5\right] \tag{2.12}$$

where

$$I_5 = \int\int_{-\infty}^{\infty}\int \frac{1}{\zeta^2\left(\zeta^2 - \frac{1}{\nu^2}\right)} e^{-ix_k\zeta_k}\, dW = 2\pi^2\nu^2\left(\frac{e^{-R/\nu} - 1}{R}\right).$$

Thus we arrive at Green's functions for concentrated couple forces as

$$G_{jn}^Y(\bar{x}) = \frac{1}{8\pi\mu}\epsilon_{njm}\nabla_m \frac{1 - e^{-R/l}}{R} \tag{2.13}$$

$$\Phi_{jn}^Y(\bar{x}) = -\frac{1}{16\pi\mu}\nabla_j\nabla_n \frac{e^{-R/l} - 1}{R} + \frac{1}{16\pi\alpha}\nabla_j\nabla_n \cdot \left(\frac{e^{-R/\nu} - e^{-R/l}}{R}\right) + \frac{\mu+\alpha}{16\pi\alpha\mu l^2}\frac{e^{-R/l}}{R}\delta_{jn} \tag{2.14}$$

3. CENTRE OF COMPRESSION AND TORSION

Let a pair of equal, opposite and collinear concentrated unit forces parallel to the x_1-axis act at the point $(x_1' + (\Delta x_1'/2), x_2', x_3')$ and $(x_1' - (\Delta x_1'/2), x_2', x_3')$. The displacement produced by these forces will be

$$^X u_j^{(1)} = \frac{G_{j1}^X\left(x_1, x_2, x_3, x_1' + \frac{\Delta x_1'}{2}, x_2', x_3'\right)}{\Delta x_1'} - \frac{G_{j1}^X\left(x_1, x_2, x_3, x_1' - \frac{\Delta x_1'}{2}, x_2', x_3'\right)}{\Delta x_1'}. \tag{3.1}$$

Letting $\Delta x_1' \to 0$, we arrive at the displacement produced by these forces as

$$^X u_j^{(1)} = \frac{\partial}{\partial x_1'} G_{j1}^X(\bar{x}, \bar{x}'). \tag{3.2}$$

Analogously, we obtain an expression for the rotation field produced by these forces as

$$^X \phi_j^{(1)} = \frac{\partial}{\partial x_1'} \Phi_{j1}^X(\bar{x}, \bar{x}'). \tag{3.3}$$

Generally, if three pairs of such forces act in the directions x_1, x_2 and x_3, the displacement and rotation field produced is given by

$$ {}^{X}u_j = \frac{\partial}{\partial x'_k} G^X_{jk} \tag{3.4} $$

$$ {}^{X}\phi_j = \frac{\partial}{\partial x'_k} \Phi^X_{jk}. \tag{3.5} $$

Using eqns (2.6) and (2.7) we arrive at

$$ {}^{X}u_j = \frac{1}{4\pi(\lambda+2\mu)} \frac{\partial}{\partial x'_j}(R^{-1}) \tag{3.6} $$

$$ {}^{X}\phi_j = 0 \tag{3.7} $$

Next, let a concentrated unit force pointing in the direction of the positive x_2-axis act at the point $(x'_1 + (\Delta x'_1/2),\ x'_2,\ x'_3)$. Also, let a similar force with opposite direction act at the point $(x'_1 - (\Delta x'_1/2),\ x'_2,\ x'_3)$. The displacement produced by these forces will be

$$ {}^{XM}u_j^{(2)} = \frac{G^X_{j2}\left(x_1, x_2, x_3, x'_1 - \frac{\Delta x'_1}{2}, x'_2, x'_3\right)}{\Delta x'_1} - \frac{G^X_{j2}\left(x_1, x_2, x_3, x'_1 + \frac{\Delta x'_1}{2}, x'_2, x'_3\right)}{\Delta x'_1}. \tag{3.8} $$

Letting $\Delta x'_1 \to 0$, one arrives at the displacement produced by this double force as

$$ {}^{XM}u_j^{(2)} = \frac{\partial}{\partial x'_1} G^X_{j2}(\bar{x}, \bar{x}'). \tag{3.9} $$

Now, let a similar pair of forces, parallel to the x_2-axis act at the points $(x'_1,\ x'_2 + (\Delta x'_2/2),\ x'_3)$ and $(x'_1,\ x'_2 - (\Delta x'_2/2),\ x'_3)$. As a result one obtains

$$ {}^{XM}u_j^{(1)} = \frac{\partial}{\partial x'_2} G^X_{j1}(\bar{x}, \bar{x}'). \tag{3.10} $$

The sum of these double forces (with moments) produces the displacement

$$ {}^{XM}u_j = -\frac{\partial}{\partial x'_1} G^X_{j2}(\bar{x}, \bar{x}') + \frac{\partial}{\partial x'_2} G^X_{j1}(\bar{x}, \bar{x}'). \tag{3.11} $$

Analogously, one can derive

$$ {}^{XM}\phi_j = -\frac{\partial}{\partial x_1} \phi^X_{j2}(\bar{x}, \bar{x}') + \frac{\partial}{\partial x_2} \phi^X_{j1}(\bar{x}, \bar{x}'). \tag{3.12} $$

Using eqns (2.6) and (2.7) there results

$$ {}^{XM}u_j = \epsilon_{ji3} \nabla'_i \left(\frac{1}{8\pi\mu} \nabla'^2 R + B \frac{e^{-R/l}}{R} \right) \tag{3.13} $$

and

$$ {}^{XM}\phi_j = \frac{1}{8\pi\mu} (\epsilon_{kj2} \nabla'_1 - \epsilon_{kj1} \nabla'_2) \frac{1 - e^{-R/l}}{R}. \tag{3.14} $$

The solution (3.13) is formally the same as that for a concentrated moment in classical elasticity theory.

Similarly one can introduce the concept of the centre of torsion for a load system consisting of three pairs of equal, opposite and collinear couple vectors acting in the directions x_1, x_2 and x_3, respectively. After an analogous procedure, bearing in mind eqns (2.10)–(2.12), we arrive at

$$ {}^{Y}u_j = 0 \tag{3.15} $$

$$ {}^{Y}\phi_j = -\frac{1}{4\pi(2\gamma+\beta)}\frac{\partial}{\partial x_j'}\left(\frac{e^{-R/\nu}}{R}\right). \tag{3.16} $$

4. THE POINT DEFECTS

Consider an infinite elastic body containing the initial distortions $\overset{0}{\gamma}_{ji}$ and $\overset{0}{\kappa}_{ji}$. If the elastic deformations produced by $\overset{0}{\gamma}_{ji}$, $\overset{0}{\kappa}_{ki}$ are denoted by $\overset{e}{\gamma}_{ji}$ and $\overset{e}{\kappa}_{ji}$, the total deformations may be written in the form

$$ \gamma_{ji} = \overset{0}{\gamma}_{ji} + \overset{e}{\gamma}_{ji}, \quad \kappa_{ji} = \overset{0}{\kappa}_{ji} + \overset{e}{\kappa}_{ji}. \tag{4.1} $$

The stress fields, σ_{ji} and μ_{ji}, are given by

$$ \sigma_{ji} = A_{ijkm}\gamma_{mk} - A_{ijkm}\overset{0}{\gamma}_{mk} \tag{4.2} $$

$$ \mu_{ji} = B_{ijkm}\kappa_{mk} - B_{ijkm}\overset{0}{\kappa}_{mk} $$

where

$$ A_{ijkm} = [\lambda\delta_{ij}\delta_{km} + (\mu+\alpha)\delta_{ik}\delta_{jm} + (\mu-\alpha)\delta_{im}\delta_{jk}] $$
$$ B_{ijkm} = [\beta\delta_{ij}\delta_{km} + (\gamma+\epsilon)\delta_{ik}\delta_{jm} + (\gamma-\epsilon)\delta_{im}\delta_{jk}]. $$

The deformations γ_{ji}, κ_{ji} may be expressed in terms of the displacements, u_i and rotations, ϕ_i

$$ \gamma_{ji} = u_{i,j} - \epsilon_{kji}\phi_k, \quad \kappa_{ji} = \phi_{i,j} \tag{4.4} $$

Substituting eqns (4.4) into (4.2) and (4.3), and using the equations of equilibrium

$$ \sigma_{ji,j} = 0 \tag{4.5} $$

$$ \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} = 0 \tag{4.6} $$

the non-homogeneous system of equations is obtained as

$$ L_{ji}u_j + R_{ji}\phi_j = A_{ijkm}\overset{0}{\gamma}_{mk,j} \tag{4.7} $$

$$ D_{ji}\phi_j + R_{ji}u_j = B_{ijkm}\overset{0}{\kappa}_{mk,j} + 2\alpha\epsilon_{ilk}\overset{0}{\gamma}_{lk} \tag{4.8} $$

where

$$ L_{ji} = A_{jpis}\nabla_p\nabla_s $$
$$ D_{ji} = B_{jpis}\nabla_p\nabla_s - 4\alpha\delta_{ij} $$
$$ R_{ji} = 2\alpha\epsilon_{jpi}\nabla_p. $$

Introduce the fictitious body forces, $\bar{X}^*$, and body couples, $\bar{Y}^*$, by

$$X_i^* = -A_{ijkm}\overset{0}{\gamma}_{mk,j} \tag{4.9}$$

$$Y_i^* = -B_{ijkm}\overset{0}{\kappa}_{mk,j} - 2\alpha\epsilon_{ilk}\overset{0}{\gamma}_{lk} \tag{4.10}$$

and investigate three particular cases:

(a) Assume $\overset{0}{\gamma}_{ji} = \overset{0}{\gamma}_{ij} = \delta_{ij}\delta_3(\bar{x})$ while $\overset{0}{\kappa}_{ji} = 0$. Then

$$X_i^* = -(3\lambda + 2\mu)\nabla_i\delta_3(\bar{x}); \quad Y_i^* = 0.$$

Substituting these relations into eqns (1.13), (1.15) and (1.16), one arrives at

$$\nabla^2\bar{u}' = \frac{3\lambda + 2\mu}{\lambda + 2\mu}\operatorname{grad}\delta(\bar{x}) \tag{4.11}$$

$$\bar{u}'' = \bar{\phi} = 0. \tag{4.12}$$

Let us also assume that the displacement field, u', has the potential, Γ. Equation (4.11) then takes the following form

$$\nabla^2\Gamma = \frac{3\lambda + 2\mu}{\lambda + 2\mu}\delta(\bar{x}) \tag{4.13}$$

from which

$$\Gamma = -\frac{3\lambda + 2\mu}{\lambda + 2\mu}\frac{1}{4\pi R}. \tag{4.14}$$

(b) Assume next that $\overset{0}{\kappa}_{ji} = \overset{0}{\kappa}_{ij} = \delta_{ij}\delta_3(\bar{x})$, while $\overset{0}{\gamma}_{ji} = 0$. In this case

$$Y_i^* = (3\beta + 2\gamma)\nabla_i\delta_3(\bar{x}); \quad X_i^* = 0$$

and substituting these relations into eqns (1.13), (1.15) and (1.16), there results

$$\bar{u}' = \bar{u}'' = 0 \tag{4.15}$$

$$D\bar{\phi} = \frac{3\beta + 2\gamma}{\beta + 2\gamma}\operatorname{grad}\delta(\bar{x}). \tag{4.16}$$

Introducing the potential, Λ by $\bar{\phi} = \operatorname{grad}\Lambda$, one obtains

$$\left(\nabla^2 - \frac{1}{\nu^2}\right)\Lambda = \frac{3\beta + 2\gamma}{\beta + 2\gamma}\delta(\bar{x}) \tag{4.17}$$

from which

$$\Lambda = -\frac{1}{4\pi\mu}\frac{3\beta + 2\gamma}{\beta + 2\gamma}\frac{e^{-R/\nu}}{R} \tag{4.18}$$

(c) Finally, case (a) can be generalized to take into account thermal distortions. For such a loading case, $\overset{0}{\gamma}_{ji} = \alpha_t\theta(\bar{x})\delta_{ij}$, $\overset{0}{\kappa}_{ji} = 0$ and

$$\bar{X}^* = -(3\lambda + 2\mu)\,\mathrm{grad}\,\theta.$$

The potential, Γ, has the form

$$\Gamma(\bar{x}) = -\frac{3\lambda + 2\mu}{4\pi(\lambda + 2\mu)}\int_V \frac{\bar{\theta}(\bar{\zeta})}{R(\bar{x}, \bar{\zeta})}\,\mathrm{d}W \tag{4.19}$$

which is known as Poisson's integral.

CONCLUSIONS

The solution to the nonhomogeneous equations of equilibrium for micropolar media is obtained by superposition of two parts, the first being the familiar solution for the Hookean case while the second part is peculiar to a Cosserat continuum. The solution has been derived using the substitution $\bar{\xi} = (1/2)\,\mathrm{curl}\,\bar{u} - \bar{\phi}$.

The solution obtained is used to derive Green's functions for the case of concentrated body forces and concentrated body couple forces. Expressions for the centres of dilatation and torsion are derived and it is shown that the first one is associated with pure displacements while the second one only with rotations. In the final section, the results derived are used to investigate three specific cases of distortion in Cosserat Media.

REFERENCES

1. H. Schaefer, Das Cosserat kontinuum. *ZAMN* **47**, 485–498 (1967).
2. N. Sandru, On some problems of the linear theory of the asymmetric elasticity. *Int. J. Engng Sci.* **4**, 81–94 (1966).
3. W. Nowacki, Green function for micropolar thermoelasticity, *Bull Acad. Polon. Sci., Series Sci. Tech.* **16**, 565–574 (1968).
4. W. Nowacki and W. K. Nowacki, The generation of waves in an infinite micropolar solid. *Proc. Vibr. Probl.* **10**, 169–191 (1969).
5. I. N. Sneddon, Fourier transforms. *London* 43 (1951).